Approximating Pi with the Golden Ratio

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Around 225 BC, Archimedes wrote the treatise *Measurement of a Circle,* which contained the first derivation of a formula for the area of a circle and the first formal approximation of the constant we now call π. At that time, it was already known that the ratio of the areas of two circles equals the ratio of the squares of their diameters. This result had appeared in Euclid's *Elements* (Euclid 1956) as Proposition 12.2. It follows that if circle O_1 has diameter d_1 and circle O_2 has diameter d_2, then

$$\frac{\text{area}(O_1)}{d_1^2} = \frac{\text{area}(O_2)}{d_2^2}.$$

From this relationship, Archimedes understood that the ratio of the area of a circle to the square of its diameter must be constant for every circle. He sought to find not only a formula for the area of a circle but also this constant ratio. To do so, Archimedes first noted that the area of a regular polygon is $ap/2$, where a is the apothem (measured from the center of the polygon to the midpoint of a side) and p is the perimeter. Using this result, Archimedes proved that the area of a circle is equal to the area of a right triangle that has one side equal to the radius r of the circle and the other side equal to the circumference C. That is, $A = rC/2$. He then found his approximation of π by using the perimeter of regular polygons, beginning with a hexagon and then doubling the number of sides to 12, 24, 48, and 96. This Archimedean method has been well outlined in many sources (see Burton 2003, Dunham 1990), and it was not improved upon until the development of calculus 1900 years later.

Today, Archimedes' method still provides a rich setting to show the interplay of algebra and geometry. It can be used not only to study π but also to introduce another geometric curiosity, called the golden ratio, or Φ. In this article, we will reemploy the Archimedean method, but we will begin with a regular pentagon rather than a hexagon. Then, by adding a little trigonometry, we will show how to find an algebraic approximation of π in terms of the golden ratio.

ARCHIMEDES' DISCOVERY OF THE "CIRCLE CONSTANT"

Archimedes knew that the ratio of the area of a circle to the square of its diameter d must be a constant k for every circle. He eventually showed that this constant is about 11/14. Using his formula for the area of a circle O, he obtained

$$k = \frac{\text{area}(O)}{d^2} = \frac{rC/2}{d^2} = \frac{dC/4}{d^2} = \frac{C}{4d},$$

which gives $C/d = 4k$.

Rather than directly finding the constant k for the ratio of area to diameter squared, Archimedes instead found an approximation of the constant $4k$. We now call this constant π, which is the ratio of circumference to diameter in any circle. To find an estimate of π, he presented the now-classical Archimedean Method of approximating the circumference of a unit circle with the perimeter p_n of an inscribed n-sided regular polygon. In particular, he gave a method for finding the side length s_{2n} of an inscribed $2n$-sided regular polygon in terms of the side length s_n of an inscribed n-sided regular polygon.

Because the side s_4 of an inscribed square and the side s_6 of an inscribed regular hexagon are easily found, we can continue to double the number of sides to generate the side lengths of the inscribed regular octagon, 16-gon, 32-gon, etc., and the side lengths of the inscribed regular 12-gon, 24-gon, 48-gon, etc. That is, we can find the side lengths of regular $4 \cdot 2^k$-sided polygons and $6 \cdot 2^k$-sided polygons that are inscribed inside a unit circle that has a diameter $d = 2$. Then using such a polygon with n sides, we can approximate π by

$$\text{(1)} \qquad \pi = \frac{C}{d} \approx \frac{p_n}{2} = \frac{n \cdot s_n}{2}.$$

By starting with both inscribed and circumscribed hexagons and doubling the number of sides four times, Archimedes used 96-sided polygons to obtain

$$\frac{C}{d} \approx \frac{211{,}875}{67{,}441} \approx 3.141635.$$

This Archimedean method of approximating π using polygon perimeters survived for centuries thereafter. In 1579, Francois Viète (1540–1603) of France found π to nine decimal places using polygons with $6 \cdot 2^{16} = 393{,}216$ sides. In 1610, Ludolph van Ceulen (1540–1610) of the Netherlands found π to 35 decimal places using polygons with $4 \cdot 2^{60}$ sides.

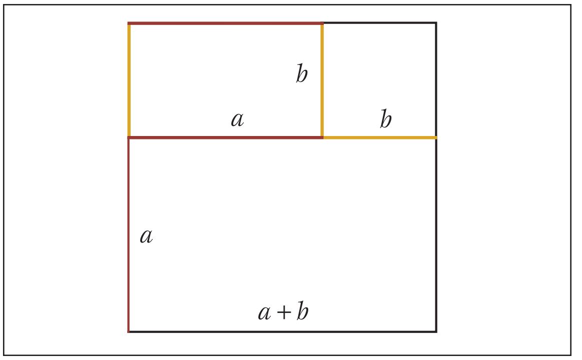

Fig. 1 Rectangles with dimensions $a \times b$ and $(a + b) \times a$

The accuracy and efficiency of approximating π with Archimedes' method has now been surpassed by methods that use calculus and infinite series. However, many lessons from geometry can still be illustrated with Archimedes' classical technique.

THE GOLDEN RECTANGLE

The golden ratio can be motivated from a "proportionally pleasing" rectangle with sides of length a and b such that the ratio $a : b$ equals the ratio $(a + b) : a$ (see **fig. 1**). In this case, what is the value of a/b?

If we have such a rectangle where $a : b = (a + b) : a$, then $a^2 = ab + b^2$, or $a^2 - ab - b^2 = 0$. Dividing by b^2, we obtain

$$a^2/b^2 - a/b - 1 = 0.$$

If we let $x = a/b$, then we obtain the "golden quadratic" equation, $x^2 - x - 1 = 0$, which has as its only positive solution

$$x = (1 + \sqrt{5})/2 \approx 1.618.$$

Thus, a/b equals $(1 + \sqrt{5})/2$. This value is called the golden ratio and is denoted by Φ.

THE GOLDEN TRIANGLE

An isosceles triangle with base angles that measure 72° also has sides that are in the golden proportion. Consider triangle ABC with $m\angle A = m\angle C = 72°$, $AB = CB = a$, and $AC = b$ (**fig. 2**). We assert that $a/b = \Phi$.

If we extend the base of the triangle ABC by length a, we obtain a new isosceles triangle DCB having equal legs of length a (**fig. 3**), and $m\angle DCB = 180° - 72° = 108°$. Thus, its congruent base angles each must measure

$$\frac{180° - 108°}{2} = 36°.$$

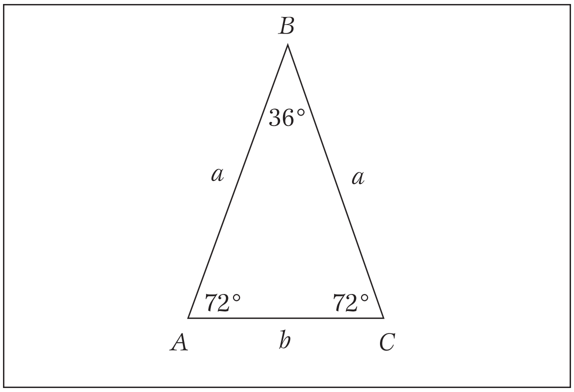

Fig. 2 The golden triangle

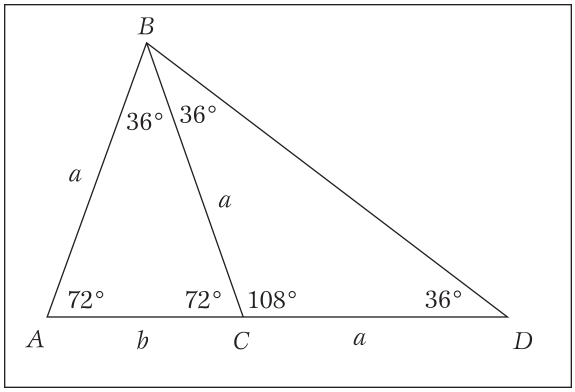

Fig. 3 Two similar golden triangles

But taking the two triangles together defines a larger triangle BDA that has base angles measuring 72° and vertex angle D measuring 36°. Thus, $\triangle BDA$ is similar to $\triangle ABC$. By this similarity, we have $a : b = BC : AC = DA : BA = (a + b) : a$, which is the ratio that defines Φ. Therefore, as with the golden rectangle, we obtain $a/b = \Phi$.

APPLYING SOME TRIGONOMETRIC IDENTITIES

Applying the law of sines and a double-angle formula to triangle ABC, we can find a trigonometric form of the golden ratio. Then we can write cos 36° and sin 36° in terms of Φ. To do so, we first recall that the law of sines gives us

$$\frac{\sin\angle A}{a} = \frac{\sin\angle B}{b} \quad \text{or} \quad \frac{\sin\angle A}{\sin\angle B} = \frac{a}{b}.$$

Then applying the formula $\sin(2\,\theta) = 2\sin\theta\cos\theta$, we obtain

$$\Phi = \frac{a}{b} = \frac{\sin 72°}{\sin 36°} = \frac{\sin(2\cdot 36°)}{\sin 36°} = \frac{2\sin 36°\cos 36°}{\sin 36°} = 2\cos 36°.$$

Thus, $$\cos 36° = \frac{\Phi}{2} = \frac{1+\sqrt{5}}{4}.$$

Because $\sin^2 36° + \cos^2 36° = 1$, we also obtain

$$(2) \quad \sin 36° = \sqrt{1-\cos^2 36°} = \sqrt{1-\left(\frac{\Phi}{2}\right)^2} = \frac{\sqrt{4-\Phi^2}}{2}.$$

THE LENGTH OF THE INITIAL INSCRIBED PENTAGON SIDE

This result allows us to find the side length of a regular pentagon inscribed in a unit circle. An inscribed regular pentagon creates five isosceles triangles that each have a vertex angle of 360°/5 = 72° opposite the side of length s_5 (**fig. 4**). By bisecting one of these triangles and then applying right-triangle trigonometry, we obtain $\sin 36° = s_5/2$. Hence,

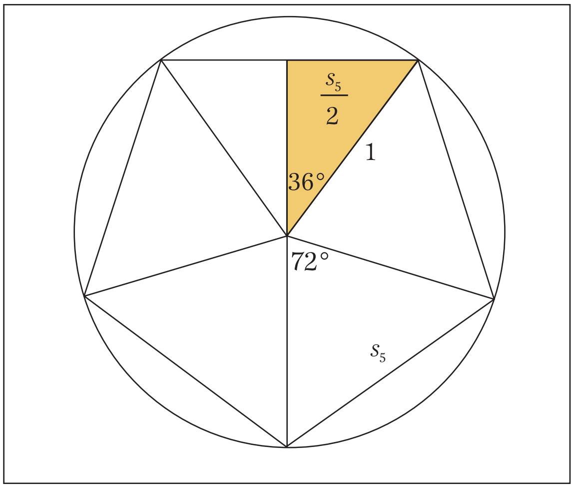

Fig. 4 Triangles inside the inscribed pentagon

by equation (2), we have

$$s_5 = 2\sin 36° = \sqrt{4-\Phi^2}.$$

FINDING THE NEXT INSCRIBED SIDE LENGTH

Next, we look at Archimedes' method for determining the new polygon side length upon doubling the number of sides. We begin by assuming that a regular n-sided polygon is inscribed inside a circle that has a radius of length 1 and that we know the length s of its sides (**fig. 5**). Then we inscribe a regular $2n$-sided polygon inside the same circle. We aim to find the length t of the new sides in terms of the original side length s (see **fig. 6**).

To find t, we first consider one isosceles triangle of the original n-sided polygon that has sides of length 1, 1, and s, and apothem of length a. The extension of the apothem to the circle creates a radius of length 1 that is a common side of two isosceles triangles within the $2n$-sided polygon. The apothem also bisects the side of length s and creates a right triangle with sides of length a, $s/2$, and 1, as shown in **figure 7**.

Applying the Pythagorean theorem to this right triangle, we obtain

$$1^2 = a^2 + \left(\frac{s}{2}\right)^2 = a^2 + \frac{1}{4}s^2.$$

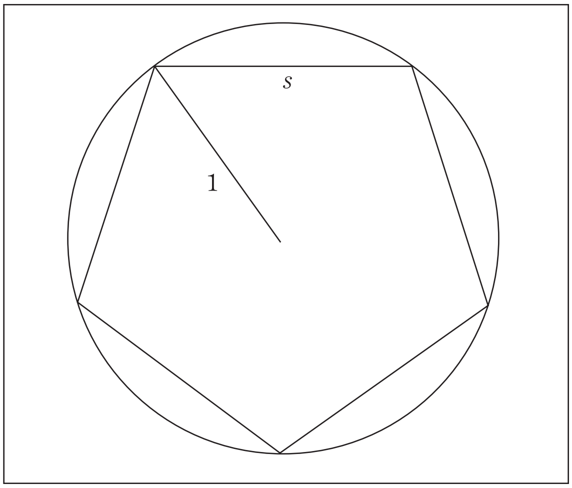

Fig. 5 An inscribed *n*-sided polygon inside a circle of radius 1

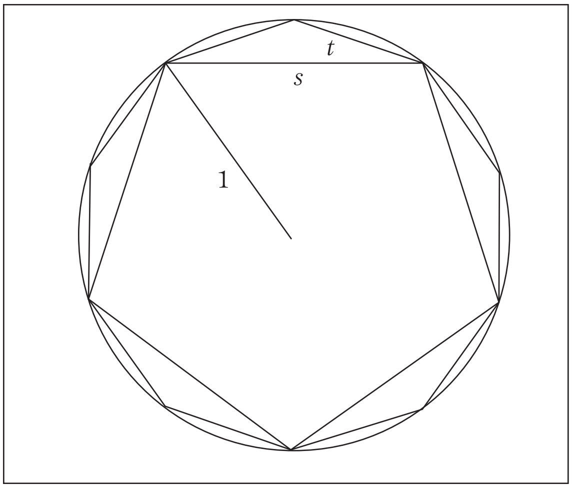

Fig. 6 An inscribed 2*n*-sided polygon inside a circle of radius 1

Then solving for *a,* we obtain

$$a = \sqrt{1 - \frac{1}{4}s^2} = \frac{\sqrt{4 - s^2}}{2}. \tag{3}$$

Therefore, $$1 - a = 1 - \frac{\sqrt{4 - s^2}}{2}.$$

Next, we use the right triangle with hypotenuse of length t and sides of length $s/2$ and $1 - a$ to obtain

$$\begin{aligned} t^2 &= \left(\frac{s}{2}\right)^2 + (1-a)^2 = \frac{1}{4}s^2 + \left(1 - \frac{\sqrt{4 - s^2}}{2}\right)^2 \\ &= \frac{1}{4}s^2 + 1 - \sqrt{4 - s^2} + \left(\frac{4 - s^2}{4}\right) \\ &= 2 - \sqrt{4 - s^2}\ . \end{aligned}$$

Hence, $$t = \sqrt{2 - \sqrt{4 - s^2}}\ .$$

More specifically, if s_n denotes the side length of a regular *n*-sided polygon that is inscribed inside a unit circle, then

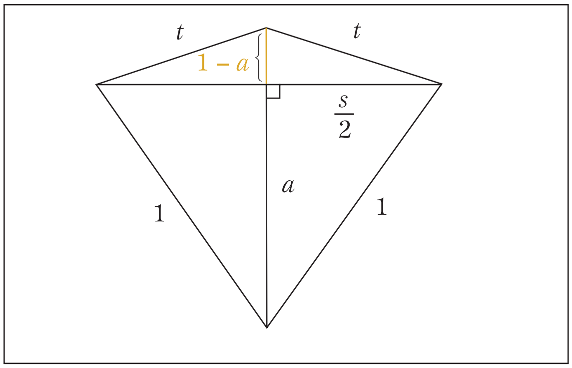

Fig. 7 One section of the inscribed polygons

$$s_{2n} = \sqrt{2 - \sqrt{4 - s_n^{\ 2}}}\ . \tag{4}$$

APPROXIMATION OF π

Because we know the side length s_5 of the initial inscribed pentagon, we can apply equation (4) repeatedly to find the side lengths of inscribed regular polygons with 10, 20, 40, 80, 160, etc., sides. In doing so, we obtain a pattern of nested radicals involving the golden ratio:

$$\begin{aligned} s_{10} &= \sqrt{2 - \sqrt{4 - s_5^{\ 2}}} \\ &= \sqrt{2 - \sqrt{4 - \left(\sqrt{4 - \Phi^2}\right)^2}} \\ &= \sqrt{2 - \Phi} \end{aligned}$$

$$\begin{aligned} s_{20} &= \sqrt{2 - \sqrt{4 - s_{10}^{\ 2}}} \\ &= \sqrt{2 - \sqrt{4 - \left(\sqrt{2 - \Phi}\right)^2}} \\ &= \sqrt{2 - \sqrt{2 + \Phi}} \end{aligned}$$

Similarly, we obtain

$$s_{40} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \Phi}}}\ ,$$

$$s_{80} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \Phi}}}}\ ,$$

and $$s_{160} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \Phi}}}}}\ .$$

Then using equation (1), we obtain our approximations of π in terms of the golden ratio Φ. See **table 1**.

The value of π to 35 decimal places is

3.14159265358979323846264338327950288.

To obtain an approximation this accurate, we would need to double the number of sides 58 times

TABLE 1

Decimal Approximations of π in Terms of Φ

n	$\pi = \frac{n \times s_n}{2}$	Decimal Approximation
5	$\frac{5\sqrt{4-\Phi^2}}{2}$	2.93892626146236564584
10	$5\sqrt{2-\Phi}$	3.09016994374947424102
20	$10\sqrt{2-\sqrt{2+\Phi}}$	3.12868930080461738020
40	$20\sqrt{2-\sqrt{2+\sqrt{2+\Phi}}}$	3.13836382911379780132
80	$40\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\Phi}}}}$	3.14078526072548872166
160	$80\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\Phi}}}}}$	3.14139079370052833368

and use an inscribed regular polygon with $5 \cdot 2^{58}$ = 1,441,151,880,758,558,720 sides. Fortunately, these values now can be generated easily with a short computer program that would have saved Van Ceulen many years of work.

USING THE AREA TO APPROXIMATE π

Archimedes discovered the area of a circle to be $A = rC/2$, which is not our modern, commonly used formula. But he also found that the ratio of circumference to diameter is constant for any circle. This constant was labeled π in 1706 by the English mathematician Sir William Jones (1675–1749). This symbol became the standard notation after the great Leonhard Euler (1707–1783) adopted it in his own work beginning in 1748 (Burton 2003). So now we say $\pi = C/d$ as well as $C = \pi d$. Substituting into Archimedes' formula for circular area, we obtain our modern formula:

$$A = \frac{1}{2}rC = \frac{1}{2}r(\pi d) = \frac{1}{2}r(\pi \cdot 2r) = \pi r^2$$

Using $r = 1$, we see that π also equals the area of a unit circle. Thus, we could use the area A_n instead of our inscribed polygons to approximate π. From Archimedes' formula, we can find this polygonal area if we know the apothem and perimeter of the polygon. But the perimeter of an n-sided regular polygon with side length s_n is simply $n \cdot s_n$, and equation (3) gives us the apothem in terms of s_n for regular polygons inscribed in a unit circle. Hence,

$$\pi \approx A_n = \frac{1}{2}ap_n = \frac{ns_n\sqrt{4-s_n^2}}{4}.$$

When we double the number of sides, we then obtain

$$(5) \qquad A_{2n} = \frac{2ns_{2n}\sqrt{4-s_{2n}^2}}{4} = \frac{ns_{2n}\sqrt{4-s_{2n}^2}}{2}.$$

But if we make the substitution

$$s_{2n} = \sqrt{2-\sqrt{4-s_n^2}},$$

then equation (5) becomes

$$A_{2n} = \frac{ns_n}{2}.$$

The details are left as an exercise. That is, the area of a regular $2n$-sided polygon inscribed in a unit circle is numerically equal to half the perimeter of a regular n-sided polygon inscribed in a unit circle. Because each value approximates π, it is one step more efficient to use the perimeter of the polygons, rather than the area, to find our approximations of π. The reader is left with the following exercises.

EXERCISES

1. Find the initial side length s_4 of a square that is inscribed in a unit circle, and then use equation (4) to find the side lengths s_8, s_{16}, s_{32}, s_{64}, and s_{128}. Then use equation (1) to find approximations of π using these polygons with $4 \cdot 2^k$ sides.

2. Find the initial side length s_6 of a regular hexagon inscribed in a unit circle. Then find the side lengths s_{12}, s_{24}, s_{48}, and s_{96}, and the resulting approximations of π obtained from using these polygons with $6 \cdot 2^k$ sides.

3. For a triangle with sides of length a, b, and c, Heron's formula gives the area as

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where $s = (a + b + c)/2$. A regular n-sided polygon inscribed in the unit circle creates n congruent triangles that have sides of length 1, 1, and s_n. Apply Heron's formula to show that the area of the polygon is given by

$$\frac{ns_n\sqrt{4-s_n^2}}{4}.$$

4. Using the fact that

$$s_{2n} = \sqrt{2-\sqrt{4-s_n^2}}$$

for regular polygons inscribed in a unit circle, show that

$$\frac{n s_{2n}\sqrt{4-s_{2n}^{2}}}{4}=\frac{n s_n}{2}.$$

5. Let S_n be the side length of a regular polygon that is circumscribed about a unit circle. For doubling the number of sides, show that

$$S_{2n}=\frac{-4+2\sqrt{4+S_n^{2}}}{S_n}.$$

6. Find S_4 for a square that is circumscribed about a unit circle, then use the result from exercise 5 to find S_8. Find the approximation of π that is obtained by using the perimeter of a circumscribed regular octagon. Repeat the process for a regular 16-sided polygon.

7. For a regular pentagon circumscribed about a unit circle, show that

$$S_5=\frac{2\sqrt{4-\Phi^{2}}}{\Phi}.$$

Hint: Use right-triangle trigonometric identities and the closed-form values of sin 36° and cos 36°.

8. Apply exercise 5 to find S_{10}, and then find the resulting approximation of π obtained by using the perimeter of a circumscribed regular decagon.

9. Find S_6 for a regular hexagon circumscribed about a unit circle. Then find S_{12} and the resulting approximation of π obtained by using the perimeter of a regular 12-sided polygon circumscribed about a unit circle.

10. Show that the area of a regular n-sided polygon circumscribed about a unit circle equals $(n \cdot S_n)/2$. In other words, with circumscribed polygons there is no difference between using the area or the perimeter to approximate π.

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300TH ANNIVERSARY OF π

In 1706, Sir William Jones labeled the familiar value of 3.141592 . . . with the Greek letter π. On March 14, join mathematics educators and students around the country in celebration of this 300th anniversary of π. Use this date to engage students in activities related to the history and concept of π and enrich and deepen the students' understanding of the concept. More information about π and about ways to mark the day can be found on the following Web sites:

archive.ncsa.uiuc.edu/edu/RSE/RSEorange/buttons.html
mathwithmrherte.com/pi_day.htm
joyofpi.com/pilinks.html
eveander.com/trivia
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mathforum.com/t2t/faq/faq.pi.html
www.eduref.org/cgi-bin/printlessons.cgi/Virtual/Lessons/Mathematics/Geometry/GEO0001.html